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DYNAMIC OPTIMIZATION OF PORTFOLIOS WITH TAIL CONDITIONAL EXPECTATION CONSTRAINTS

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Abstract. We consider an optimal portfolio problem subject to Tail Conditional Expectation (TCE) constraint. This problem is formulated as a constrained optimal stochastic control problem with or without consumption; where the financial market is composed of d-risky assets driven by geometric Brownian motion and one riskless asset. Explicit optimal strategies are derived using a combination of Hamilton-Jacobi-Bellman (HJB) equation and the Lagrange multiplier. We deduced from our observations that the constraint imposed, reduces risky investment and that the TCE-constrained optimal investment is a multiple of the market price of risk and its volatility which is known as the relative risk tolerance in a risky markets.

1. INTRODUCTION

Markowitz pioneered the optimal portfolio problem based on the mean-variance approach where variance was substituted as a risk measure [10]. It is basically a single period model which makes a one-off decision at the beginning of the period and holds on until the end of the period. Later, [9] considered the continuous-time optimal portfolio models and stochastic control theory was employed to find the optimal strategies in environment where risks are controlled indirectly via the value function.

Since the dawn of financial history, risk measurement has preoccupied financial market investors as well as the institutions involved. The search for a better risk measure has proven to be impractically complex. To this end, Value-at-Risk (VaR), a downside risk measure emerged as one of the most popular tools in measuring risk with regulatory authorities enforcing its use [1, 6, and 14]. The Value-at-Risk (VaR) is the maximum loss which can be expected at a given confidence level over a given time horizon.

The conditional expectation of X given that $X > VaR^{\alpha,x}$, denoted by

$$TCE^{\alpha,x} = E\left(\left(X \,/\, X \geq VaR^{\alpha,x}\right)\right)$$

at confidence level $1-\alpha$ is called the Tail Conditional Expectation (TCE).

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Both VaR and TCE are important measures of risks frequently encountered in the insurance and financial investment [12].Despite its widespread acceptance, VaR is known to possess unappealing features. [2] proposed an axiomatic foundation for risk measures; by identifying four properties that a reasonable risk measure should satisfy and provided a characterization of the risk measures satisfying these properties, which they called coherent risk measures. By these axioms, VaR is not coherent, as it does not satisfy the subadditivity property [1]. Tail Conditional Expectation (TCE), on the other hand, for continuous distribution is a coherent risk measure [5].

This paper is focused on dynamic portfolio problem with the aim of finding the optimal investment and consumption strategies. In the existing literature, investment and consumption strategies have received a considerable attention from various authors and are often studied in separate problems. This problem has been considered by [1] in a non-closed form solution. Here, we consider stochastic control problem with investment and with/without intermediate consumption. In addition, direct constrained market risk is also considered, which is asymmetric since individuals are naturally risk averse. We apply the TCE constraint while maximizing the investor's logarithmic utility over consumption throughout the investment horizon and over the terminal wealth. We derived explicit optimal strategies using a combination of HJB-equation and the method of Lagrange multiplier.

The paper is organized as follows: Section 2 gives the financial model of the market and description of the dynamics of asset portfolio. Section 3 introduces the control processes, Tail Conditional Expectation (TCE) risk measure used in this paper. In section 4, the optimization problem is stated and the optimal strategies obtained. Section 5 summarizes the paper with derivation of optimal conditions.

2. FINANCIAL MODEL AND ITS FORMULATIONS

We consider a Black-Scholes type of financial market with the following properties see [8]. Uncertainty is represented by a complete filtered probability space $(\Omega, f, (f_t), P)$ and throughout the paper, we denote by $(f_t)_{t\geq 0}$ the neutral filtration i.e. $f_t = \sigma\{W(s);, 0 \leq s \leq t\}$ where $(W(.))$ is a standard d-dimensional Brownian motion defined on this space with values in R^d. The market consists of one riskless bond $S_o(t)$ and several risky assets $S_i(t)$ on the interval $[0, T]$. Their respective prices $(S_0(t))_{0\leq t\leq T}$ and $(S_i(t))_{0\leq t\prec T}$ for $i = 1,..., d$ evolve according to the equations:

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1 \quad , \qquad (1)$$

$$dS_i(t) = S_i(t)d\xi_i(t), \quad S_i(0) = s_i > 0 \qquad (2)$$

where

$$d\xi_i(t) = \mu_i dt + \sum_{j=i}^{d} \sigma_{ij} dW_j(t), \quad \xi_i(0) = 0, \; t \in [0, T].$$

For $d \in N$, we denote by $W(t) = (W_1(t), ..., W_d(t))^T$ a standard d-dimensional Brownian motion, i.e., a vector of d independent one-dimensional Brownian process, r the riskless interest rate, $\mu = (\mu_1, ..., \mu_d)^T$ the vector of stock-appreciation rates and $\sigma = (\sigma_{ij})_{1 \le i, j \le d}$ the matrix of stock volatilities. The symbol $(^T)$ denotes transpose. Again, we assume that all coefficients in (1)-(2) are deterministic functions and satisfy:

$$\int_0^T \left(|r|^2 + |\mu| + |\sigma|^2 \right) dt < \infty \qquad (3)$$

Let $\pi_i(t)$ be an admissible portfolio process, i.e. $\pi_i(t)$ is the fraction of the wealth invested into stock i at time t and the remaining fraction $1 - \sum_{j=1}^{d} \pi_i(t)$ is invested into bond. A non-negative, adapted process $(c(t))t > 0$ satisfying for the investment horizon $T > 0$ $\int_0^T c(t)dt < \infty \; a.s$ is called the consumption rate process [7]. The independent economic agent invests according to an investment strategy that can be described by the $(d+1)$-dimensional, predictable process

$$h(t) = (h_0(t), ..., h_d(t)), \qquad (4)$$

where $h_i(t), \; i = 1, ..., d$ denotes the number of stocks i held in the portfolio at time $t, (i = 0 \; refers \; to \; the \; bond)$. The process h describes an economic investor's portfolio as carried forward through time. The value of the investor's wealth at time t is represented as

$$X_h(t) = h_0(t) S_0(t) + \sum_{i=1}^{d} h_i(t) S_i(t). \qquad (5)$$

Since $h_0(t)$ and $h_i(t)$ represent the amount invested into bond and stock respectively, and as we work in self-financing portfolios, we have (5) rewritten as

$$X^A(t) = 1 - \sum_{j=1}^{d} \pi_i(t) S_0(t) + \sum_{i=1}^{d} \pi_i(t) S_i(t) \qquad (6)$$

such that in differential terms (6) becomes

$$dX^A(t) = X^A(t) \left[\left(r_t + \sigma \pi^T(t) \lambda \right) dt + \sigma \pi^T(t) dW(t) \right] - c(t)dt$$

$$t > 0, \quad X^A(0) = x, \qquad (7)$$

where $A = (\pi, c)$ is the control process, $\lambda = \sigma^{-1}(\mu - rI)$. By putting $v(t) = \dfrac{c(t)}{X^A(t)}$ which implies that $c(t) = v(t) X^A(t)$ and we can rewrite the differential of the wealth process as:

$$dX(t) = X \left[\left(r + \sigma \pi^T(t) \lambda - v(t) \right) dt + \sigma \pi^T(t) dW(t) \right]. \qquad (8)$$

3. CONTROL PROCESSES AND TAIL CONDITIONAL EXPECTATION (TCE) RISK MEASURE

Definition 1: Let $t \in [0,T]$ be a fixed financial horizon. A stochastic control process $A = (A(t))t \geq 0 = (\pi(t), v(t)), t \geq 0$ is called admissible if it is $(f_t)_{t\geq 0}$ adapted with values in $\Re^d x \Re^+$ for which equation (8) has a unique strong **a.s**. positive solution $(X^A(t)), t \geq 0$ satisfying

$$E\int_0^T \left((\pi(u))^2 + v(u)\right)du < \infty. \qquad (9)$$

We denote by A the class of all admissible control processes. By ito's formula, for every $A \in A(0)$, equation (8) has the solution:

$$X_t^A = x \exp\left(\int_0^T a(u)du + \int_0^T \sigma\pi^T(u)dW(u)\right),\ u \geq 0 \qquad (10)$$

where

$$a(u) = r - v + \sigma\pi^T\lambda - \frac{1}{2}\left(\sigma\pi^T\right)^2,\ u \geq 0$$

.

Definition 2: Given some probability level $\alpha \in (0,1)$ and given a time horizon, $\tau > 0$, the Value-at-Risk (VaR) of time of a portfolio $X^A(t)$ at a confidence level $1-\alpha$ is given by

$$\begin{aligned} VaR^{\alpha,x}(t) &= \inf\left\{L \geq 0 : P\left(X^A(t) - X^A(t+\tau) \geq L \middle| f_t\right) < \alpha\right\} \\ &= -Q^{\alpha,x}(t) \qquad (11) \end{aligned}$$

where

$$Q^{\alpha,x}(t) = Sup\left\{\begin{matrix} L \in \Re^d : P\left(X^A(t+\tau) - X^A(t) \leq L \middle| f_t\right) \\ \leq \alpha \end{matrix}\right\}$$

is the quantile of the projected wealth surplus at the horizon τ [6]. VaR_t^{α} is therefore the loss wealth at the horizon τ which could be exceeded only with a small conditional probability α if the current portfolio were kept unchanged. The fact $VaR^{\alpha,x}(t)$ is computed under the assumption that the current portfolio is kept unchanged reflects the actual practice and the fact that the financial institutions monitoring their investment do not typically know the investors' future portfolio choices over VaR horizon. The Measure of VaR in (11) only requires the knowledge of the current portfolio value, the current asset and the conditional distribution of asset returns.

Proposition 1

$$VaR^{\alpha,x}(t) = -Q^{\alpha,x}(t) =$$

$$X^{A}(t)\left[1-\exp\left(\left(\left(r+\sigma\pi^{T}(t)\lambda-v(t)-\frac{1}{2}\left(\sigma\pi^{T}(t)\right)^{2}\right)\tau+\Phi^{-1}(\alpha)\sigma\pi^{T}(t)\sqrt{\tau}\right)\right)\right]^{+} \quad (12)$$

where $x^{+}=\max[0,x]$, $\Phi(x)$ and $\Phi^{-1}(x)$ denote the normal distribution and inverse distribution functions respectively.

Proof: see [3].

Definition 3: The $TCE^{\alpha,x}(t)$ at confidence level $1-\alpha$ is defined as:

$$TCE^{\alpha,x}(t) = \mathrm{E}\left\{X^{A}(t)\middle|X^{A}(t+\tau)\geq VaR^{\alpha,x}(t)\right\}$$

$$=\frac{1}{\alpha}E\left\{\left(X^{A}(t)-X^{A}(t+\tau)\right)\middle|\begin{matrix}\left(X^{A}(t)-X^{A}(t+\tau)\right)\geq\\ -Q^{\alpha}(t)\end{matrix}\right\}^{+} \quad (13)$$

where $\left|(A)\right.$ is the indicator function of the set A. In other words, the Tail Conditional Expectation of wealth $X^{A}(t)$ at time t is the conditional expected value of the loss exceeding $-Q^{\alpha}(t)$. Again, given the log-normal distribution of asset returns, the TCE can be explicitly computed as seen below.

Proposition 2: The computation of the Tail conditional Expectation is [3];

$$TCE^{\alpha,x}(t) = X\left[\alpha-\exp\left(\begin{matrix}\left(r+\sigma\pi^{T}(t)\lambda-v\right)\tau.\\ \Phi\left(\Phi^{-1}(\alpha)-\sigma\pi^{T}(t)\sqrt{\tau}\right)\end{matrix}\right)\right]^{+} \quad (14)$$

4. STATEMENT OF OPTIMAL PROBLEMS

Now we consider the problem of an investor who starts with an endowment $X(t)$ and must select an optimal investment and consumption to maximize (over all admissible $(\pi(t),v(t))$), the expected discounted logarithmic utility of terminal wealth at time T and consumption over the entire horizon $[0,T]$. The $TCE^{\alpha,x}(t)$ risk measure is no larger than some pre-specified level $\rho^{\alpha}(X,t)\geq 0$ where $\rho^{\alpha}(X,t)\geq 0$ is thus defined as some risk bound.

In mathematical terms, the final stochastic optimal control problem with TCE constraint is:

$$\underset{(\pi,v)\in A}{Max}\ E_{t}\left\{\int_{0}^{T}e^{-\delta(s-t)}Log\left(c(s)\right)ds+e^{-\delta(T-t)}Log\left(X(T)\right)\right\} \quad (15)$$

subject to the wealth differential

$$dX(t) = X\left[\left(r + \sigma\pi^{T}(t)\lambda - v(t)\right)dt + \sigma\pi^{T}(t)dW(t)\right] \quad (16)$$

and the TCE constraint

$$TCE^{\alpha,x}(t) \le \rho^{\alpha}(X,t)\ \forall\ t \in [0,T]. \quad (17)$$

Here, E_t denotes the expectation operator at time t given the wealth process with respect to the admissible control strategies. The value function is logarithmic which yields to very clear explicit results. $\delta > 0$ is the rate of discount of consumption and terminal wealth [13].

Remark 1

The expression for TCE in (14) implies that a portfolio satisfies the constraint $TCE^{\alpha,x}(t) \le \rho^{\alpha}(X,t)\ \forall\ t \in [0,T]$ if and only if:

$$Log\left(\alpha - \frac{\rho^{\alpha}(X,t)}{X}\right) - \left(r + \sigma\pi^{T}\lambda - v\right)\tau - Log\ \Phi\left(\Phi^{-1}(\alpha) - \sigma\pi^{T}\sqrt{\tau}\right) \le 0 \quad (18)$$

.

It can be verified that (18) is equivalent to:

$$Log\left(\alpha - \frac{\rho^{\alpha}(X,t)}{X}\right) - \left(r + \sigma\pi^{T}\lambda - \frac{1}{2}\left(\sigma\pi^{T}\right)^{2} - v\right)\tau + \frac{1}{2}\Phi^{-2}(\alpha) - \Phi^{-1}(\alpha)\sigma\pi^{T}\sqrt{\tau} \le 0. \quad (19)$$

We see that (19) is quadratic and satisfies an upper and lower bound on the fraction $\pi(t)$ allocated to the risky asset which is

$$\pi^{-}(x,t) \le \pi(t) \le \pi^{+}(x,t).$$

Hence,

$$\pi^{\pm}(x,t) = \frac{\lambda\sqrt{\tau} \pm \Phi^{-1}(\alpha) \pm Z}{\sigma\sqrt{\tau}} \quad (20)$$

,

where

$$Z = \sqrt{\left(\lambda\sqrt{\tau} \pm \Phi^{-1}(\alpha)\right)^{2} - 2\left(Log\left(\alpha - \frac{\rho^{\alpha}}{X}\right)^{+} - r\tau + v\tau + \frac{1}{2}\Phi^{-2}(\alpha)\right)}.$$

We therefore rewrite the stochastic control problem in equations (15) – (17) as

$$\underset{(\pi,\nu)\in A}{Max}\ E_t\left\{\int_0^T e^{-\delta(s-t)}Log\ (c(s))ds\ +\ e^{-\delta(T-t)}Log\ (X(T))\right\} \quad (21)$$

s.t

$$dX\ (t) = X\left[\left(r+\sigma\pi^{\ T}(t)\lambda-\nu(t)\right)dt\ +\ \sigma\pi^{\ T}(t)dW\ (t)\right] \quad (22)$$

with the constraint

$$Log\left(\alpha-\frac{\rho^{\alpha}(X,t)}{X}\right)-\left(r+\sigma\pi^{\ T}\lambda-\frac{1}{2}\left(\sigma\pi^{\ T}\right)^2-\nu\right)\tau + \frac{1}{2}\Phi^{-2}(\alpha)-\Phi^{-1}(\alpha)\sigma\pi^{\ T}\sqrt{\tau}\ \le 0. \quad (23)$$

In what follows we state;

THEOREM 1

Let $J(X,t)$ denote the value function for the stochastic control problem of (21) – (23) and let

$$\beta^{\pm}\ =\ \frac{\left|k\right|\sqrt{\tau}\ \pm\ \Phi^{-1}(\alpha)\pm Z}{\left|k\right|\sqrt{\tau}} \quad (24)$$

where $k\ =\sigma\left(\sigma\sigma^{\ T}\right)^{-1}\lambda$, $\theta\ =\nu$, then $\beta^{+}(X,t)\ge 0$ for all $(X,t)\in(0,\infty)x[0,T]$ and J solves the HJB-equation

$$\delta J\ =\begin{cases} J_t+\ J_x xr\ -\ \dfrac{1}{2}\dfrac{J_x^{\ 2}}{J_{xx}}\left|k\right|^2\ -Log\ \ J_x^{-1}-1 \\ \\ if\ \ -\dfrac{J_x}{xJ_{\ xx}}\le\beta_{\alpha}^{+},\ \dfrac{J_x^{-1}}{x}\le\ \theta \\ \\ J_t+\ J_x x\left(r+\left|k\right|^2\beta_{\alpha}^{+}\right)+\dfrac{1}{2}J_{xx}x^2\left|k\beta_{\alpha}^{+}\right|^2 \\ -Log\ \ \theta\ -1 \qquad\qquad otherwise \end{cases} \quad (25)$$

with terminal condition as

$$J(X,T)=\ Log\ \ X_T. \quad (26)$$

letting

$$\beta(X,t)=\ \min\ \left[-\frac{J_x(X,t)}{xJ_{\ xx}(X,t)},\ \beta^{+}(X,t)\right], \qquad \theta\ =\ \frac{1}{xJ_{\ x}(X,t)} \quad (27)$$

the optimal investment strategy becomes

$$\pi(X,t)=\ \beta(X,t)\left(\sigma\sigma^{\ T}\right)^{-1}\lambda \quad (28)$$

and the optimal consumption given as

$$\nu(X,t)=\ \theta(X,t) \quad (29)$$

so that

$$c(X,t) = \theta(X,t)X \qquad (29a)$$

solves (21) – (23).

Proof

In applying the dynamic programming approach, we solve the HJB equation associated with the logarithmic utility maximization problem of equation (21). Defining the value function

$$J(X,t) = \underset{(\pi,c)\in A}{Sup}\; E\left\{\int_0^T e^{-\delta(s-t)} Log\,(c_s)ds + e^{-\delta(T-t)} Log\left(X_T^A\right)\right\} \qquad (30)$$

Following [11], we deduce the corresponding HJB equation as:

$$\delta J = \max_{\nu\geq 0,\,\pi\in\Re^d} \left\{ \begin{array}{l} J_x\left(x\left(r+\sigma\pi^T\lambda\right)\right) + J_t + \frac{1}{2}J_{xx}x^2\left|\sigma\pi^T\right|^2 + \\ Log\,(\nu x) - J_x\nu x \end{array} \right\} \qquad (31)$$

subject to the terminal condition

$$J(X,T) = Log\; X_T. \qquad (32)$$

where the subscripts on J denote the derivatives and $x = X^A(t)$ the wealth at time t.

In solving the HJB-equation (31), subject to the Tail Conditional Expectation (TCE) constraint of (23), we reduce the HJB-equation (31) to a non-linear Partial Differential Equation (PDE) of J only. We therefore apply the method of Lagrange function to impose the risk measure as:

$$\begin{array}{l} L(\pi,\nu,\psi) = J_x\left(x\left(r+\sigma\pi^T\lambda\right)\right) + J_t + \frac{1}{2}J_{xx}x^2\left(\sigma\pi^T\right)^2 + \\ Log\,(\nu x) - J_x\nu x - \psi\left(TCE_t^{\alpha} \leq \rho^{\alpha}(x,t)\right) \end{array} \qquad (33)$$

so that by substitution of (23) for the TCE constraint, we have:

$$\begin{array}{l} L(\pi,\nu,\psi) = J_x\left(x\left(r+\sigma\pi'\lambda\right)\right) + J_t + \frac{1}{2}J_{xx}x^2\left(\sigma\pi^T\right)^2 + \\ Log\,(\nu x) - J_x\nu x - \psi\left\{ \begin{array}{l} Log\left(\alpha - \frac{\rho^{\alpha}(X,t)}{X}\right) - \\ \left(r+\sigma\pi^T\lambda - \frac{1}{2}\left(\sigma\pi^T\right)^2 - \nu\right)\tau + \\ \frac{1}{2}\Phi^{-2}(\alpha) - \Phi^{-1}(\alpha)\sigma\pi^T\sqrt{\tau} \leq 0 \end{array} \right\}. \end{array} \qquad (34)$$

Where $\psi = \psi(x,t)$ is the Lagrange multiplier. We therefore derive the respective first-order conditions with respect to admissible controls and Lagrange multiplier of the static optimization of (31) as:

$$\frac{\partial L}{\partial \pi} = J_x x\lambda + J_{xx} x^2 \sigma\sigma^T \pi + \psi\left[\left(\lambda - \sigma\sigma^T \pi\right)\tau - \Phi^{-1}(\alpha)\frac{\sigma\sigma^T \pi}{\sigma\pi^T}\sqrt{\tau}\right]. \quad (35)$$

$$\frac{\partial L}{\partial \nu} = 0 = \frac{1}{\nu} - J_x x - \psi\tau \quad . \quad (36)$$

And

$$Log\left(\alpha - \frac{\rho^{\alpha}(X,t)}{X}\right) - \left(r + \sigma\pi^T \lambda - \frac{1}{2}\left(\sigma\pi^T\right)^2 - \nu\right)\tau + \frac{1}{2}\Phi^{-2}(\alpha) - \Phi^{-1}(\alpha)\sigma\pi^T\sqrt{\tau} = 0. \quad (37)$$

$$\frac{\partial L}{\partial \psi} = H(x,t) = -Log\left(\alpha - \frac{\rho^{\alpha}(X,t)}{X}\right) - \left(r + \sigma\pi^T \lambda - \frac{1}{2}\left(\sigma\pi^T\right)^2 - \nu\right)\tau + \frac{1}{2}\Phi^{-2}(\alpha) - \Phi^{-1}(\alpha)\sigma\pi^T\sqrt{\tau} \geq 0, \quad (38)$$

where (37) is called complementary slackness condition.

Rearranging equation (35) gives

$$\left[J_{xx} x^2 - \psi\left(\tau - \Phi^{-1}(\alpha)\frac{\sqrt{\tau}}{\sigma\pi^T}\right)\right]\pi = -\left[J_x x + \psi\tau\right]\left(\sigma\sigma^T\right)^{-1}\lambda. \quad (39)$$

Since the terms in square brackets are scalar functions of (x,t), this implies that (24) must hold for some scalar function β [5]. Replacing the optimal strategies of equations (28)and(29with equation (38) gives

$$Log\left(\alpha - \frac{\rho^{\alpha}(X,t)}{X}\right) - \left(r_t + (\beta - \frac{1}{2}\beta^2)|k|^2\right)\tau - \theta\tau + \frac{1}{2}\Phi^{-2}(\alpha) - \Phi^{-1}(\alpha)|\beta k|\sqrt{\tau} \leq 0 \quad (40)$$

which is equivalent to:

$$\beta_{\alpha}^{-} \leq \beta \leq \beta_{\alpha}^{+},$$

where $\beta^{\pm}$ is as defined in equation (24) and $\theta = \frac{1}{xJ_x}$.

The complementary slackness condition is given as:

$$\psi(x,t)H(x,t) = 0 \quad for \quad \psi(x,t) \geq 0.$$

By this, equations (35) and (36) and the complementary slackness condition (37) implies that:

$$\left(J_{xx} x^2 \beta + J_x x\right)\lambda = 0 \quad (41)$$

$$\beta(x,t) = -\frac{J_x}{xJ_{xx}} \qquad (42)$$

$$\theta(x,t) = \frac{1}{xJ_x} \quad . \qquad (43)$$

So that optimal strategies satisfies

$$\pi(x,t) = \beta(x,t)\left(\sigma\sigma^T\right)^{-1}\lambda \qquad (44)$$

$$\nu(x,t) = \theta(x,t) . \qquad (45)$$

Substituting the first order conditions into (31) gives the partial differential equation below

$$\delta J = J_t + rxJ_x - \frac{1}{2}\frac{(J_x)^2}{J_{xx}}|\lambda|^2 - LogJ_x^{-1} - 1. \qquad (46)$$

If this PDE above has a solution $J(X,t)$ such that the strategies defined by (44) and (45) are feasible, then we know from the verification theorem (see, [4]) that this strategy is indeed the optimal investment and consumption strategies and the function $J(X,t)$ is indeed the value function [10]. Problems with no utility from consumption i.e. $Logc = 0$ is of course optimal not to consume and it is relatively easy to see that the last two terms in the RHS of (46) will vanish and the equation simplifies to:

$$\delta J = J_t + rxJ_x - \frac{1}{2}\frac{(J_x)^2}{J_{xx}}|\lambda|^2. \qquad (47)$$

OBSERVATIONS

From the analysis of the model above, and to further appreciate the implications of TCE constraints for optimal strategies, we consider below alternative or equivalent specification of the function $\rho^{\alpha}(X,t)$ which identifies the maximum admissible TCE of risk by the investor at any time $t \in [0,T]$. Notice that equivalent constraint on TCE for logarithmic utility is binding iff:

$$\log\left(\alpha - \rho^{\alpha}(x,t)\right) \leq \inf_{0\leq t\leq T} \leq \beta_{\alpha}^{+}(x,t),$$

where $\rho^{\alpha} > 0$ is as before defined as some risk bound.

Moreover, it follows from (44) that the TCE-constrained optimal investment is a multiple of the market price of risk and its volatility. Following from the terminology of [10], we refer to this multiple as the relative risk tolerance.

5. CONCLUSION

In this paper, we considered an optimal portfolio problem subject to TCE constraint using a combination HJB-equation and Lagrange Multiplier with a Logarithmic utility function. We derived explicit expressions for the optimal portfolio choice (investment) and consumption given that the complementary slackness condition

$\psi(x,t)H(x,t)=0$ *for* $\psi(x,t)\geq 0$. We deduced from our observations that the constraint imposed such that $\beta_{\alpha}^{-}\leq\beta\leq\beta_{\alpha}^{+}$, reduces risky investment and that the TCE-constrained optimal investment is a multiple of the market price of risk and its volatility which is known as the relative risk tolerance in a risky markets. Also, we deduce that the optimal consumption strategy is to consume a time-varying fraction of wealth as seen from the theorem.

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